About the gradually movement of a liquid sphere in a viscous media

Note de
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Introduction

The problem, which we refer to in the following[1] is to be seen as a generalization of the well-known solution of the problem of hydrodynamics G. G. Stokes solved in 1851[2] where we still keep Stokes postulate that the movement of the sphere is so slow that we can neglect the effect of inertia in relation to viscous forces and suppose that the sphere is no more a rigid body of revolution but is instead made out of a viscous liquid.
Because the position of the acting forces is important on a liquid sphere we suppose that the movement of the sphere is due to a difference in density and therefore due to the force of gravity.
In this form the problem is not only of theoretical but also of practical importance, because it enables us, e.g., to correct for the error we make when we compute the charge of an electron caused by the velocity of decent of mist particles.

[1] At the suggestion of Herrn Prof. M. Smoluchowski
[2] G. G. Stokes: On the effect of the internal friction of fluids on the motion of pendulums. In: *Transactions of the Cambridge Philosophical Society*, Vol. 9, 1851, pp. 8–106.

1 Theoretical Considerations

We follow the well-known fundamental formulæ of hydrodynamics which in our case are given as:
for the inner liquid

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \mu \cdot \nabla^2 u - \sigma g \\ \frac{\partial p}{\partial y} &= \mu \cdot \nabla^2 v \\ \frac{\partial p}{\partial z} &= \mu \cdot \nabla^2 w \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \tag{1a}$$

for the outer liquid

$$\left.\begin{aligned} \frac{\partial p'}{\partial x} &= \mu' \cdot \nabla^2 u' - \sigma' g \\ \frac{\partial p'}{\partial y} &= \mu' \cdot \nabla^2 v' \\ \frac{\partial p'}{\partial z} &= \mu' \cdot \nabla^2 w' \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 \end{aligned}\right\} \tag{1b}$$

where p denotes the pressure, u, v, w the components of velocity, σ the density of the liquid, μ the index of friction and the primed symbols are referred to the outer liquid accordingly.
Let us now assume that the position of the sphere in space keeps unaltered and the outer liquid moves with velocity U at infinite distance in relation to the sphere. We therefore have at infinity:

$$u' = U\,, \quad v' = 0\,, \quad w' = 0\,. \tag{2}$$

The conditions for the boundary surface between both liquids are:

$$u = u'\,, \quad v = v'\,, \quad w = w' \tag{3}$$

and owing to the dynamic equilibria every component of force acting on an element of the boundary surface must be equal (see Lamb, *Hydrodynamics* p. 537)[3].

[3]The difference between the limiting cases given here and those of Duhem are only apparently and are based on the designation of the values $p_{xx}, \ldots$ Lamb subsumes in these values the hydrostatic pressure p whereas Duhem designates it with Π (cf. Duhem, *Recherches sur l'Hydrodynamique*).

The general solution of Equation 1a and Equation 1b, adapted to spherical surfaces is given by Lamb (*Hydr.* p. 550) in the form:

$$u = \frac{1}{\mu} \cdot \sum \left\{ \frac{r^2}{2 \cdot (2n+1)} \cdot \frac{\partial P_n}{\partial x} + \frac{n \cdot r^{2n+3}}{(n+1) \cdot (2n+1) \cdot (2n+3)} \cdot \frac{\partial}{\partial x} \frac{P_n}{r^{2n+1}} \right\} + \sum \left(\frac{\partial \Phi_n}{\partial x} + z \frac{\partial \chi_n}{\partial y} - y \frac{\partial \chi_n}{\partial z} \right) \tag{4}$$

and similarly by performing a cyclic permutation for the equations containing v, w, where P_n, Φ_n, χ_n are spherical harmonics of degree n and P_n in our case is $(p + \sigma g x)$.
From the condition that the movement of the sphere is symmetrical with respect to the x-axis, it follows that:

$$\chi = 0\,. \tag{5}$$

We try the following *Ansatz*:

$$\begin{aligned} P'_{-2} &= A \cdot \frac{x}{r^3}\,, \\ \Phi'_1 &= U \cdot x\,, \\ \Phi'_{-2} &= B \cdot \frac{x}{r^3}\,, \end{aligned}$$

which results in

$$\left.\begin{aligned} u' &= \left(\frac{A}{2\mu' r^3} - \frac{3B}{r^5} \right) \cdot x^2 + \frac{1}{2} \cdot \frac{A}{\mu' r} + \frac{B}{r^3} + U \\ v' &= \left(\frac{A}{2\mu' r^3} - \frac{3B}{r^5} \right) \cdot xy \\ w' &= \left(\frac{A}{2\mu' r^3} - \frac{3B}{r^5} \right) \cdot xz\,. \end{aligned}\right\} \tag{6}$$

From this it follows for the velocity normal to the surface of the sphere:

$$\frac{xu' + yv' + zw'}{r} = \left(\frac{A}{\mu' r} - \frac{2B}{r^3} + U \right) \cdot \frac{x}{r} \tag{7}$$

Now let us suppose that the sphere retains its morphology (what is proved to be true by the following calculation) from what follows that the term in brackets must vanish for $r = a$, i.e.:

$$\left(\frac{A}{\mu' a} - \frac{2B}{a^3} + U \right) = 0\,. \tag{8}$$

For the inner liquid we set:

$$\Phi_1 = C \cdot x\,, \quad P_1 = E \cdot x\,.$$

From the condition that on the surface of the sphere the normal component of velocity must vanish, it follows that:

$$C = -\frac{a^2}{10} \cdot Ex \,.$$

From this we get:

$$\left.\begin{aligned} u &= -Hx^2 + 2Hr^2 - Ha^2 \,, \\ v &= -Hxy \,, \\ w &= -Hxz \,, \quad \text{where} \quad H = \frac{E}{10\mu} \,. \end{aligned}\right\} \tag{9}$$

Recognizing Equation 6, Equation 8, and Equation 9, it follows from the limiting conditions according to Equation 3 that:

$$\left.\begin{aligned} A &= Ha^3\mu' - \frac{3Ua\mu'}{2} \,, \\ B &= \frac{Ha^5}{2} - \frac{Ua^3}{4} \,. \end{aligned}\right\} \tag{10}$$

The limiting condition of equal pressure at the boundary surface between the two liquids is stated in the usual form as:

$$p_{hx} = p'_{hx} \,, \quad p_{hy} = p'_{hy} \,, \quad p_{hz} = p'_{hz} \,, \tag{11}$$

where

$$\begin{aligned} p_{hx} &= lp_{xx} + mp_{xy} + np_{xz} \,, \\ p_{hy} &= lp_{yx} + mp_{yy} + np_{yz} \,, \\ p_{hz} &= lp_{zx} + mp_{zy} + np_{zz} \,. \end{aligned}$$

Similarly equations hold for the primed p.
Because of the well-know reason how these pressures p_{xx}, p_{xy}, ... are determined in a viscous liquid we get:
for the inner liquid

$$\left.\begin{aligned} p_{hx} &= -\frac{9\mu Hx^2}{a} + \frac{\sigma g x^2}{a} + 3\mu Ha \\ p_{hy} &= -\frac{9\mu Hxy}{a} + \frac{\sigma gxy}{a} \\ p_{hz} &= -\frac{9\mu Hxz}{a} + \frac{\sigma gxz}{a} \end{aligned}\right\} \tag{12a}$$

for the outer liquid

$$\left.\begin{aligned} p'_{hx} &= \frac{6H\mu' x^2}{a} + \frac{\sigma' g x^2}{a} - 3H\mu' a + \frac{3}{2}\cdot\frac{U\mu'}{a} \\ p'_{hy} &= \frac{6H\mu' xy}{a} + \frac{\sigma' g xy}{a} \\ p'_{hz} &= \frac{6H\mu' xz}{a} + \frac{\sigma' g xz}{a}\,, \end{aligned}\right\} \tag{12b}$$

and by comparison of these relations according to Equation 11 the following values are obtained for:

$$H = \frac{(\sigma - \sigma')\cdot g}{\mu'\cdot(9\lambda + 6)}\,, \tag{13a}$$

$$U = \frac{2}{9}\cdot\frac{(\sigma - \sigma')}{\mu'}\cdot g a^2 \cdot \frac{3\lambda + 3}{3\lambda + 2}\,, \tag{13b}$$

$$A = -\frac{(\sigma - \sigma')\cdot a^3 g}{3}\,, \tag{13c}$$

$$B = -\frac{(\sigma - \sigma')\cdot a^5 g\lambda}{2\cdot(9\lambda + 6)\cdot\mu'}\,, \tag{13d}$$

where λ is an abbreviation for $\frac{\mu}{\mu'}$.
In that all conditions of our assumptions are fulfilled, the problem is finally solved and as a reverse conclusion it is proofed that the assumption of a retained spherical boundary surface is true.
Above all the value for the velocity of the sphere according to Equation 13b is of special interest.
In general a rigid sphere is moving at the slowest velocity possible because in the case that $\lambda = \infty$, Equation 13b takes on the form of Stokes' law.

$$U = \frac{2}{9}\cdot\frac{(\sigma - \sigma')}{\mu'}\cdot g a^2\,.$$

Otherwise even larger velocities as those calculated from Stokes formula are obtained, e.g. the velocity of a water drop falling in air is 1,284.000 $\cdot a^2$, which is about 0.3 % larger as the one calculated after Stokes and with an air bubble in water the difference is even 50 %.
To get a clearer picture of the movement of the liquids we use a geometric illustration of the appropriate streamlines.
If we compute the stream function ψ, which equals the amount of liquid that flows

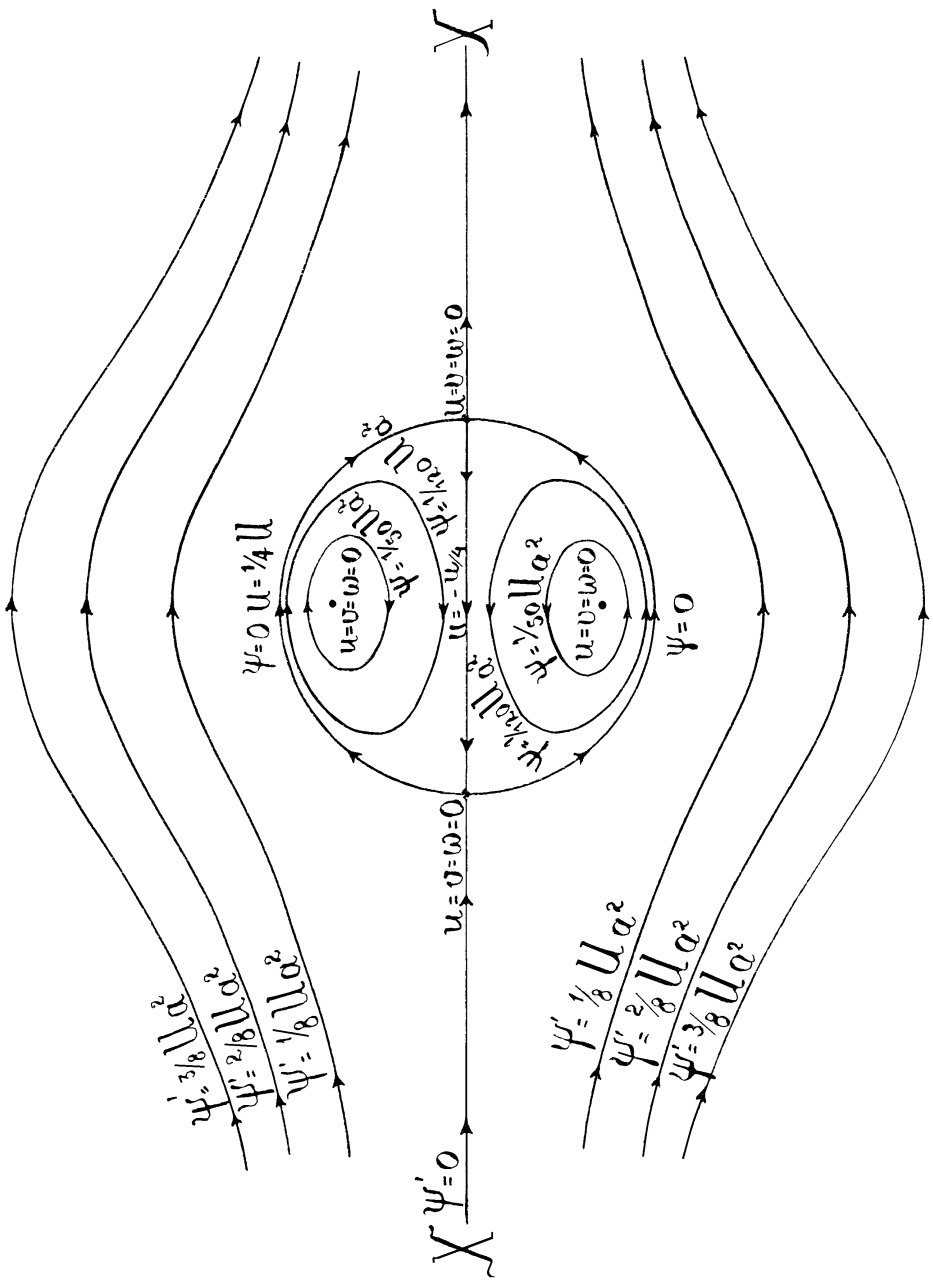

Figure 1: Streamlines around a liquid sphere

through a circular area perpendicular to the x-axis according to the equation

$$u = -\frac{1}{\varrho} \cdot \frac{\partial \psi}{\partial \varrho} \,,$$
$$\bar{v} = \frac{1}{\varrho} \cdot \frac{\partial \psi}{\partial x} \,,$$

where u and $\bar{v}$ are the velocity components in the direction of x and ϱ those perpendicular to the latter, and if we further assume e.g. that $\mu = \mu'$.
Then we have:
for the inner liquid

$$u = -\frac{Ux^2}{4a^2} + \frac{Ur^2}{2a^2} - \frac{U}{4} \,,$$
$$\bar{v} = -\frac{Ux\varrho}{4a^2} \,,$$
$$\psi = \frac{1}{8}\varrho^2 \frac{U}{a^2} \cdot \left(a^2 - r^2\right) \,;$$

for the outer liquid

$$u' = -\frac{5}{8}\frac{Ua}{r^3} \cdot \left(1 - \frac{3}{5} \cdot \frac{a^2}{r^2}\right) \cdot x^2 + U \cdot \left(1 - \frac{5}{8} \cdot \frac{a}{r} - \frac{1}{8} \cdot \frac{a^3}{r^3}\right) \,,$$
$$\bar{v}' = -\frac{5}{8}\frac{Ua}{r^3} \cdot \left(1 - \frac{3}{5} \cdot \frac{a^2}{r^2}\right) \cdot x\varrho \,,$$
$$\psi' = -\frac{1}{2}\varrho^2 U \cdot \left(1 - \frac{5}{4} \cdot \frac{a}{r} + \frac{1}{4} \cdot \frac{a^3}{r^3}\right) \,;$$

for $r = a$ it follows that $\psi = \psi' = 0$, so the interface boundary of the liquid-liquid interface is made up of streamlines.
The streamlines illustrated in Figure 1 are generated through a cut of the two surfaces $\psi =$ constant and the meridian plane.
The work due to gravity and the energy, which is dissipated by friction of both liquids while they move against each other, must of course be equal, which I also have shown to be true through an explicit calculation of providing Rayleigh's dissipation function.